On the determinateness of semi-infinite bimatrix games

Jurg, P.; Tijs, S.H.

Published in:
International Journal of Game Theory

Publication date:
1993

International Journal of Game Theory (1993) 21: 361-369

Game Theory

On the Determinateness of Semi-Infinite Bimatrix Games[1]

PETER JURG

Department of Mathematics, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands

STEF TIJS

Department of Mathematics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands

Abstract: In this paper it is shown that every $2 \times \infty$ bimatrix game is weakly determined. For the proof of this fact we introduce ε-optimal and k-guaranteeing points for a convex set in $\mathbb{R}^m$ and a labeling technique which is typical for $2 \times \infty$ bimatrix games.

Key words: Game Theory, semi-infinite bimatrix game, weak determinateness, convex set, ε-optimal in nonnegative direction, k-guaranteeing in nonnegative direction, labeling.

1 Introduction

An important role in non-cooperative game theory is played by *equilibria,* i.e. outcomes from which unilateral deviation yields no profitable effect. A game which possesses an equilibrium is called *determined.* In case a game does not possess an equilibrium, *almost equilibria* become important. In Lucchetti, Patrone and Tijs (1986) different types of almost equilibria were defined for two-person games which were used for the notion of *weak determinateness* of these games.

Following the terminology of Lucchetti, Patrone and Tijs, von Neumann (1928) showed that every finite matrix game is determined and Wald (1945) showed that every bounded semi-infinite matrix game is weakly determined. Tijs (1975) proved that also unbounded semi-infinite matrix games are weakly determined.

For finite bimatrix games determinateness was shown in Nash (1951). For semi-infinite ($m \times \infty$) bimatrix games, where the second player has an upper bounded payoff matrix, weak determinateness was shown in Tijs (1975), Tijs (1977) and Tijs (1981). In Lucchetti et al. (1986) was dealt with semi-infinite bimatrix games with various boundedness restrictions guaranteeing weak determinateness. But the problem of weak determinateness of general semi-infinite bimatrix games is still unsolved.

In section 4 we show weak determinateness of general $2 \times \infty$ bimatrix games. The proof partly relies on a labeling technique which is inspired by Borm, Gijsberts and Tijs (1988). The other tools we use in our proof are results concerning notions of optimal k-guaranteeing and ε-dominating elements of a (not necessarily bounded) convex set in $\mathbb{R}^n$, described in section 3. In section 2 we recall the definitions of

[1] The authors are indebted to Peter Borm, Jos Potters and an unknown referee for useful comments.

0020-7276/93/4/361-369 \$ 2.50

weak determinateness and some of the results of Lucchetti et al. (1986). The paper concludes with some remarks in section 5.

Notation. For $x, y \in \mathbb{R}^m$ we denote $x \geq y$ if $x_i \geq y_i$ for all $i \in \{1, \ldots, m\}$ and $x \cdot y := \sum_{i=1}^{m} x_i y_i$. For a set $V \subset \mathbb{R}^m$ we denote by $\mathrm{conv}(V)$ the set of all convex combinations of finitely many elements of V. Further $e_1, e_2, \ldots, e_m$ are the standard unit vectors in $\mathbb{R}^m$ and $1_m \in \mathbb{R}^m$ denotes the vector with every coordinate equal to 1.

2 Weak Determinateness of Semi-Infinite Bimatrix Games

Let $A := [a_{ij}]_{i=1}^{m}{}_{j=1}^{\infty}$ and $B := [b_{ij}]_{i=1}^{m}{}_{j=1}^{\infty}$ be two real $m \times \infty$ matrices. By (A, B) we denote the two-person non-cooperative (bimatrix) game consisting of the payoff matrices A and B for player I and player II respectively and strategy spaces

$$\Delta_m := \left\{ x \in \mathbb{R}^m \mid \sum_{i=1}^{m} x_i = 1, x \geq 0 \right\}$$

for player I and

$$\Delta_\infty := \left\{ (y_1, y_2, \ldots) \mid y_j \in \mathbb{R}, y_j \geq 0 \text{ for } j \in \mathbb{N}, \sum_{j=1}^{\infty} y_j = 1 \text{ and } y_j = 0 \text{ if } j \text{ is large} \right\}$$

for player II. In a play of this game player I chooses a *strategy* $p \in \Delta_m$ and player II chooses a strategy $q \in \Delta_\infty$. Subsequently player I obtains a *payoff* $pAq := \sum_{i=1}^{m} \sum_{j=1}^{\infty} p_i a_{ij} q_j$ and player II obtains $pBq := \sum_{i=1}^{m} \sum_{j=1}^{\infty} p_i b_{ij} q_j$.

We recall the definitions of the four types of *almost equilibria* for this semi-infinite bimatrix game occurring in Lucchetti et al. (1986). Let $k_1, k_2 \in \mathbb{R}$ and $\varepsilon_1, \varepsilon_2 \geq 0$. Then $(\hat{p}, \hat{q}) \in \Delta_m \times \Delta_\infty$ is called

an $(\varepsilon_1, \varepsilon_2)$-*equilibrium* if

$$\hat{p} A \hat{q} + \varepsilon_1 \geq p A \hat{q} \quad \text{for all } p \in \Delta_m \tag{1}$$

$$\hat{p} B \hat{q} + \varepsilon_2 \geq \hat{p} B q \quad \text{for all } q \in \Delta_\infty \tag{2}$$

an (ε_1, k_2)-*equilibrium* if (1) holds and

$$\hat{p} B \hat{q} \geq k_2 \tag{3}$$

a (k_1, ε_2)-*equilibrium* if (2) holds and

$$\hat{p} A \hat{q} \geq k_1 \tag{4}$$

a (k_1, k_2)*-equilibrium* if both (3) and (4) hold.

By $E^{\varepsilon_1,\varepsilon_2}(A, B)$, $E^{\varepsilon_1,k_2}(A, B)$, $E^{k_1,\varepsilon_2}(A, B)$ and $E^{k_1,k_2}(A, B)$ we denote the sets of $(\varepsilon_1, \varepsilon_2)$-, (ε_1, k_2)-, (k_1, ε_2)- and (k_1, k_2)-equilibria respectively.

The game (A, B) is called *weakly determined* if it has one of the following properties:

(WD1) $E^{\varepsilon_1,\varepsilon_2}(A, B) \neq \emptyset$ for all $\varepsilon_1, \varepsilon_2 > 0$
(WD2) $E^{k_1,\varepsilon_2}(A, B) \neq \emptyset$ for all $k_1 \in \mathbb{R}$ and $\varepsilon_2 > 0$
(WD3) $E^{\varepsilon_1,k_2}(A, B) \neq \emptyset$ for all $\varepsilon_1 > 0$ and $k_2 \in \mathbb{R}$
(WD4) $E^{k_1,k_2}(A, B) \neq \emptyset$ for all $k_1, k_2 \in \mathbb{R}$.

Since the equilibrium conditions are equal to (1) and (2) with $\varepsilon_1 = \varepsilon_2 = 0$, determinateness of (A, B) implies (WD1).

The converse need not hold as the following example shows.

Example 2.1. Consider the $2 \times \infty$ bimatrix game (A, B) where $A := \begin{bmatrix} 0 & 1 & 2 & 3 & \dots \\ 1 & 0 & 0 & 0 & \dots \end{bmatrix}$ and $B := \begin{bmatrix} 1 & -1 & -2 & -3 & \dots \\ 1 & 1 & 2 & 3 & \dots \end{bmatrix}$. We show that this game has the property (WD1). Let $\varepsilon_1, \varepsilon_2 > 0$ and take $\ell \in \mathbb{N}$ such that $\ell \geq \dfrac{1}{\varepsilon_2}$. Define $\hat{p} := (\tfrac{1}{2}, \tfrac{1}{2}) \in \Delta_2$ and $\hat{q} \in \Delta_\infty$ such that $\hat{q}_1 = \dfrac{\ell - 1}{\ell}$ and $\hat{q}_\ell = \dfrac{1}{\ell}$. Then $e_1 A \hat{q} = e_2 A \hat{q}$ so that $(\hat{p}, \hat{q})$ satisfies (1). Moreover $\hat{p} B \hat{q} = \dfrac{\ell - 1}{\ell} \geq 1 - \varepsilon_2 \geq \sup_{q \in \Delta_\infty} \hat{p} B q - \varepsilon_2$. Hence $(\hat{p}, \hat{q})$ also satisfies (2).

Now assume $(\hat{p}, \hat{q}) \in \Delta_m \times \Delta_\infty$ satisfies (1) and (2) for this game with $\varepsilon_1 = \varepsilon_2 = 0$. For $\hat{p} \in \operatorname{conv}(\{(\tfrac{1}{2}, \tfrac{1}{2}), e_2\}) \setminus \{(\tfrac{1}{2}, \tfrac{1}{2})\}$ we can make $\hat{p} B q$ for $q \in \Delta_\infty$ arbitrary large by letting $q_j = 1$ for j large. So $\hat{p} \in \operatorname{conv}(\{e_1, (\tfrac{1}{2}, \tfrac{1}{2})\})$. Then $\sup_{q \in \Delta_\infty} \hat{p} B q$ is attained uniquely by $\hat{q} \in \Delta_\infty$ with $\hat{q}_1 = 1$. However, then $e_1 A \hat{q} < e_2 A \hat{q}$, which implies $\hat{p} = e_2 \notin \operatorname{conv}(\{e_1, (\tfrac{1}{2}, \tfrac{1}{2})\})$. Since this is a contradiction, (A, B) is not determined.

An $m \times \infty$ matrix $D := [d_{ij}]_{i=1}^{m}\,{}_{j=1}^{\infty}$ is called *upper bounded* (*lower bounded*) if there exists a $k \in \mathbb{R}$ such that $d_{ij} \leq k$ $(d_{ij} \geq k)$ for all $i \in \{1, \dots, m\}$ and $j \in \mathbb{N}$.

The following theorem summarizes some results of Lucchetti et al. (1986).

Theorem 2.1. Let (A, B) be an $m \times \infty$ bimatrix game. Then (A, B) is weakly determined if one of the following assertions hold:

(i) B is upper bounded
(ii) both A and B are lower bounded
(iii) B is lower bounded and A is not upper bounded
(iv) $m = 2$ and B is upper or lower bounded.

The $2 \times \infty$ game of example 2.1 does not satisfy condition (iv) of theorem 2.1, since the matrix B is neither upper nor lower bounded. Nevertheless this game has the property (WD1). In section 4 we show that also in general the boundedness conditions on the matrix B in theorem 2.1 (iv) can be omitted.

3 Tools

Let C be a convex set in $\mathbb{R}^m$ and let Δ_m be as in section 2. Here we interpret Δ_m as the set of all (normalized) nonnegative directions in $\mathbb{R}^m$. We say that C is upper bounded in the direction $p\in\Delta_m$ if $\sup_{x\in C} p\cdot x<\infty$. The set of all nonnegative directions in which C is upper bounded is denoted by $UB(C)$, and its complement, $\Delta_m\backslash UB(C)$, by $NUB(C)$. Note that $UB(C)$ is convex.

Let $p\in UB(C)$ and $\varepsilon\geq 0$. An element $\bar{x}$ of C is called *ε-optimal in direction p* if $p\bar{x}+\varepsilon\geq \sup_{x\in C} p\cdot x$. The set of all elements of C that are ε-optimal in direction p is non-empty and convex. We denote it by $O_{p,\varepsilon}$.

Now let $p\in NUB(C)$ and $k\in\mathbb{R}$. An element $\bar{x}$ of C is called *k-guaranteeing in direction p* if $p\bar{x}\geq k$. The set of all elements of C that are k-guaranteeing in direction p is denoted by $G_{p,k}$. Also this set is non-empty and convex.

Note that for $p\in UB(C)$ and $\varepsilon_1\geq\varepsilon_2\geq 0$ we have that $O_{p,\varepsilon_1}\supset O_{p,\varepsilon_2}$ and for $p\in NUB(C)$ and $k_1, k_2\in\mathbb{R}$ with $k_1\geq k_2$ we have that $G_{p,k_1}\subset G_{p,k_2}$.

We investigate these notions in the following example.

Example 3.1. We let B be as in example 2.1 and let C be the set of all convex combinations of finitely many columns of B, i.e. $C=\{Bq\mid q\in\Delta_\infty\}$. Then $C\subset\mathbb{R}^2$ is the grey area in figure 3.1. We have $UB(C)=\text{conv}(\{e_1, (\frac{1}{2},\frac{1}{2})\})$ and $NUB(C)=\text{conv}(\{(\frac{1}{2},\frac{1}{2}), e_2\})\backslash\{(\frac{1}{2},\frac{1}{2})\}$. In figure 3.1 $\hat{p}\in UB(C)$ and $O_{\hat{p},\varepsilon}$ is the vertically shaded part of C. Further $G_{e_2,k}$ is the horizontally shaded part of C. Note that $O_{\hat{p},\varepsilon}\cap G_{e_2,k}=\emptyset$. One easily checks that $O_{\left(\frac{1}{2},\frac{1}{2}\right),\varepsilon}\cap G_{e_2,k}=\emptyset$ for all $\varepsilon>0$ and $k\in\mathbb{R}$.

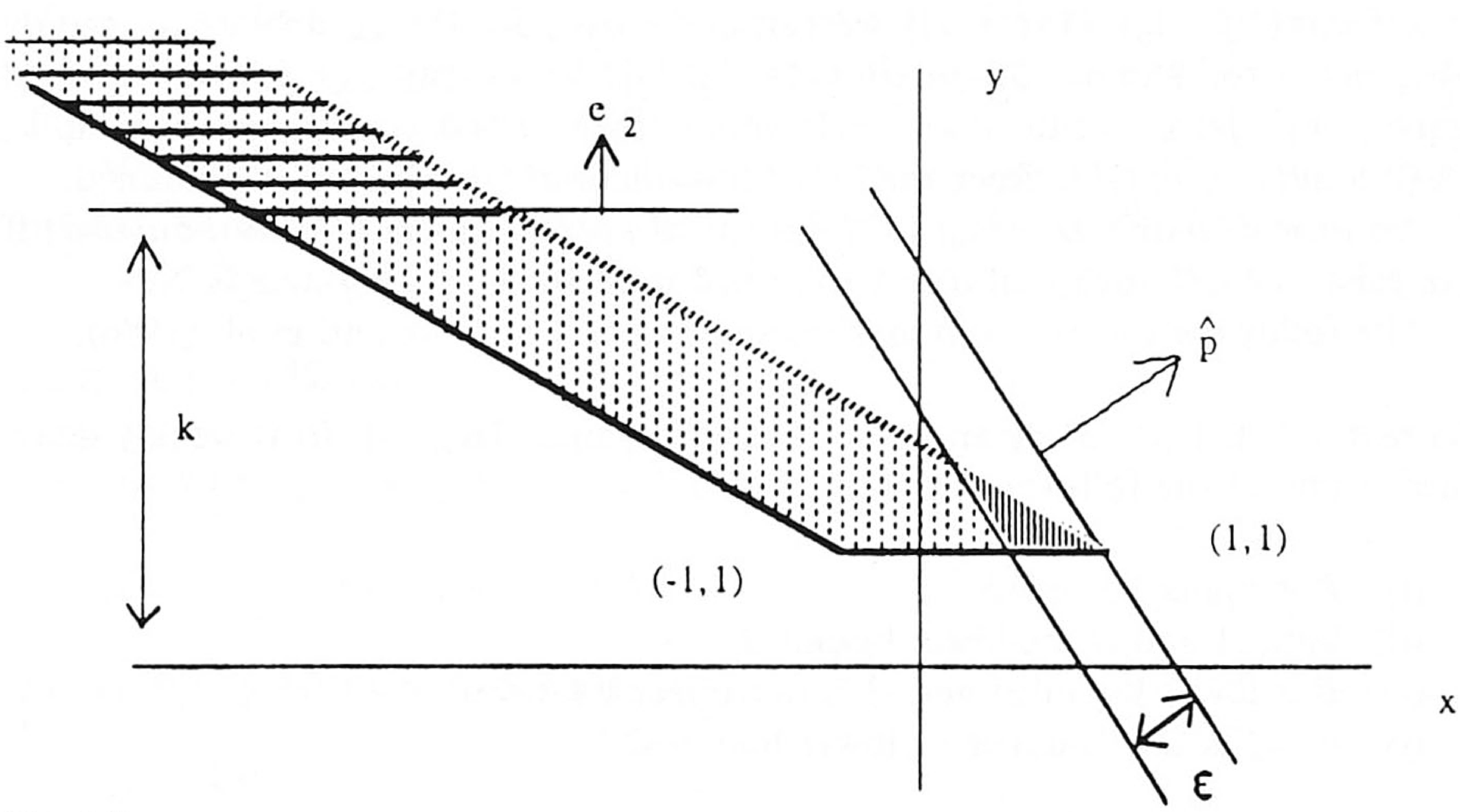

Fig. 3.1

Proposition 3.1 tells about non-empty intersection in general. For the proof of this proposition we need the following well-known lemma.

Lemma 3.1. Let C be a convex set in $\mathbb{R}^m$ and let $P := \{y \in \mathbb{R}^m \mid a^i \cdot y \geq b^i, i \in \{1, \ldots, t\}\}$ with $a^1, \ldots, a^t \in \mathbb{R}^m$ and $b^1, \ldots, b^t \in \mathbb{R}$. Suppose $P \neq \emptyset$ and $C \cap P = \emptyset$. Then there is an $a \in \operatorname{conv}(\{a^1, \ldots, a^t\})$ such that $a \cdot x \leq a \cdot y$ for all $x \in C$ and all $y \in P$.

Proof. Note that P is convex. From a well-known separation theorem for convex sets we obtain the existence of an $\tilde{a} \in \mathbb{R}^m \setminus \{0\}$ such that $\tilde{a} \cdot x \leq \tilde{a} \cdot y$ for all $x \in C$ and $y \in P$. We have to show that there is a $\lambda \in \mathbb{R}^t$, $\lambda \geq 0$, such that $\sum_{i=1}^t \lambda_i a^i = \tilde{a}$. Suppose such a λ does not exist. Then Farkas' lemma implies the existence of a $y^0 \in \mathbb{R}^m$ such that $\tilde{a} \cdot y^0 < 0$ and $a^i \cdot y^0 \geq 0$ for all $i \in \{1, \ldots, t\}$. The latter inequalities imply that for a positive $\mu \in \mathbb{R}$ and a $y \in P$ also $y + \mu y^0 \in P$. Then for a fixed $x^0 \in C$ the first inequality yields $\tilde{a} \cdot (y + \mu y^0) = \tilde{a} \cdot y + \mu \tilde{a} \cdot y^0 < \tilde{a} \cdot x^0$ for large μ. This contradicts $\tilde{a} \cdot (y + \mu y^0) \geq \tilde{a} \cdot x^0$. □

Proposition 3.1. Let C be a convex set in R^m and suppose $UB(C)$ is non-empty. Then for every $p \in NUB(C)$, $k \in \mathbb{R}$ and $\varepsilon > 0$ there exists a $p' \in UB(C)$ such that $G_{p,k} \cap O_{p',\varepsilon} \neq \emptyset$.

Proof. Take $p \in NUB(C)$, $k \in \mathbb{R}$ and $\varepsilon > 0$. It suffices to show that $G_{p,k} \cap (\bigcup_{p' \in UB(C)} O_{p',\varepsilon}) \neq \emptyset$. For a moment suppose that this intersection is empty and take $x^0 \in G_{p,k}$. Then we show a contradiction in three steps.

(a). First we show that we can find an $x^1 \in C$ such that $x^1 \geq x^0 + \varepsilon 1_m$. Assume such an x^1 does not exist. Then $C \cap \{y \in \mathbb{R}^m \mid y \geq x^0 + \varepsilon 1_m\} = \emptyset$. Applying lemma 3.1 with $a^i = e_i$ and $b^i = x_i^0 + \varepsilon$ for $i = 1, \ldots, m$, we find a $\bar{p} \in \operatorname{conv}(\{e_1, \ldots, e_m\}) = \Delta_m$ such that $\bar{p} \cdot x \leq \bar{p} \cdot y$ for all $x \in C$ and all $y \in \mathbb{R}^m$ satisfying $y \geq x^0 + \varepsilon 1_m$. In particular this implies $\bar{p} \cdot x \leq \bar{p} \cdot x^0 + \varepsilon$ for all $x \in C$. Hence $\bar{p} \in UB(C)$ and $x^0 \in O_{\bar{p},\varepsilon}$. However, this contradicts the assumption of the empty intersection.

(b). By part (a) we can find an $x^1 \in C$ such that $x^1 \geq x^0 + \varepsilon 1_m$. This implies $p \cdot x^1 \geq p \cdot x^0 + \varepsilon$ and consequently $x^1 \in G_{p,k}$. Then, by repetition of the arguments above, we can construct a sequence $x^0, x^1, x^2, \ldots$ of elements of $G_{p,k}$ such that $x^n \geq x^{n-1} + \varepsilon 1_m \geq x^0 + n\varepsilon 1_m$ for $n \in \mathbb{N}$.

(c). We use part (b) to show a contradiction. Take $\tilde{p} \in UB(C)$. Then for large n we have $\tilde{p} \cdot x^n \geq \tilde{p} \cdot x^0 + n\varepsilon > \sup_{x \in C} \tilde{p} \cdot x$, which is a contradiction. □

Roughly speaking proposition 3.1 relates for a convex set C in $\mathbb{R}^m$ the sets $UB(C)$ and $NUB(C)$. For our purposes (cf. section 4) this relation suffices. Now we focus on $UB(C)$. First we give a definition.

Let C be a convex set in $\mathbb{R}^m$ and let $\varepsilon > 0$. A subset S of C is said to *ε-dominate* C if for every $x \in C$ there is a $y \in S$ such that $y \geq x - \varepsilon 1_m$.

Note that if $UB(C) = \emptyset$, there is no such subset for any $\varepsilon > 0$.

Lemma 3.2. (Tijs, Refs. 6 and 7). Let C be a convex set in $\mathbb{R}^m$ and suppose $\Delta_m = UB(C)$. Then for every $\varepsilon > 0$ there exists a finite subset S of C such that S ε-dominates C.

Proposition 3.2. Let C be a convex set in $\mathbb{R}^m$ and suppose $UB(C) \neq \emptyset$. Let $\mathscr{P} := \mathrm{conv}(\{p^1, \ldots, p^\ell\})$ with $p^1, \ldots, p^\ell \in UB(C)$. Then for every $\varepsilon > 0$ there is a finite subset S of C such that $S \cap O_{p,\varepsilon} \neq \emptyset$ for every $p \in \mathscr{P}$.

Proof. Let $\varepsilon > 0$ and $\bar{C} := \{(p^1 \cdot x, p^2 \cdot x, \ldots, p^\ell \cdot x) \mid x \in C\}$. Then $\bar{C}$ is a convex set in $\mathbb{R}^\ell$ and since for any $\bar{p} \in \Delta_\ell$ we have $\sup_{\bar{x} \in \bar{C}} \bar{p} \cdot \bar{x} = \sup_{x \in C} \sum_{i=1}^{\ell} \bar{p}_i (p^i \cdot x) < \infty$, we have $UB(\bar{C}) = \Delta_\ell$. Applying lemma 3.2 to $\bar{C}$ and $\frac{1}{2}\varepsilon$ we obtain a finite subset $\bar{S}$ of $\bar{C}$ such that $\bar{S}$ $\frac{1}{2}\varepsilon$-dominates $\bar{C}$. Let S be a finite subset of C such that $\bar{S} = \{(p^1 \cdot x, p^2 \cdot x, \ldots, p^\ell \cdot x) \mid x \in S\}$. Take an arbitrary $p \in \mathscr{P}$ and $\hat{x} \in O_{p, \frac{1}{2}\varepsilon}$. Then there is an $\bar{s} \in \bar{S}$ such that $\bar{s} \geq (p^1 \cdot \hat{x}, p^2 \cdot \hat{x}, \ldots, p^\ell \cdot \hat{x}) - \frac{1}{2}\varepsilon 1_\ell$, or equivalently, there is an $s \in S$ such that $p^i \cdot s \geq p^i \cdot \hat{x} - \frac{1}{2}\varepsilon$ for all $i \in \{1, \ldots, \ell\}$. Then also $p \cdot s \geq p \cdot \hat{x} - \frac{1}{2}\varepsilon$. Since $\hat{x} \in O_{p, \frac{1}{2}\varepsilon}$, this implies $p \cdot s \geq \sup_{x \in C} p \cdot x - \frac{1}{2}\varepsilon - \frac{1}{2}\varepsilon = \sup_{x \in C} p \cdot x - \varepsilon$. Hence $s \in O_{p,\varepsilon}$. This completes the proof. □

4 Determinateness of $2 \times \infty$ Bimatrix Games

In this section (A, B) is a $2 \times \infty$ bimatrix game and C is the set of all convex combinations of finitely many columns of B, i.e. $C = \{Bq \mid q \in \Delta_\infty\} \subset \mathbb{R}^2$.

We define a labeling function $\lambda : \Delta_2 \to \{0, 1, 2\}$ as follows: For $p \in UB(C)$:

$$\lambda(p) = \begin{cases} 1 & \text{if there is an } \varepsilon > 0 \text{ such that } e_1 A q > e_2 A q \text{ for all } q \in \Delta_\infty \text{ with } Bq \in O_{p,\varepsilon}. \\ 2 & \text{if there is an } \varepsilon > 0 \text{ such that } e_1 A q < e_2 A q \text{ for all } q \in \Delta_\infty \text{ with } Bq \in O_{p,\varepsilon}. \\ 0 & \text{else} \end{cases}$$

and for $p \in NUB(C)$:

$$\lambda(p) = \begin{cases} 1 & \text{if there is a } k \in \mathbb{R} \text{ such that } e_1 A q > e_2 A q \text{ for all } q \in \Delta_\infty \text{ with } Bq \in G_{p,k}. \\ 2 & \text{if there is a } k \in \mathbb{R} \text{ such that } e_1 A q < e_2 A q \text{ for all } q \in \Delta_\infty \text{ with } Bq \in G_{p,k}. \\ 0 & \text{else} \end{cases}$$

Accordingly we call (A, B) *0-determined* if there is a $p \in \Delta_2$ with $\lambda(p) = 0$, *1-determined* if $\lambda(p) = 1$ for all $p \in \Delta_2$, *2-determined* if $\lambda(p) = 2$ for all $p \in \Delta_2$ and *quasi 0-determined* if $UB(C) \neq \emptyset$ and for every $\varepsilon > 0$ there is a pair $(p, q) \in UB(C) \times \Delta_\infty$ satisfying $Bq \in O_{p,\varepsilon}$ such that $e_1 A q = e_2 A q$.

Note that if (A, B) is *not* quasi 0-determined, then a (uniform) ε_0 exists such that for all pairs $(p, q) \in UB(C) \times \Delta_\infty$ satisfying $Bq \in O_{p,\varepsilon_0}$, $e_1 A q \neq e_2 A q$. We use this in the proof of lemma 4.2.

It is the aim of this section to show that every $2 \times \infty$ bimatrix game is 0-, 1-, 2- or quasi 0-determined. This ensures weak determinateness since

Proposition 4.1. For a $2\times\infty$ bimatrix game 0-, 1-, 2- and quasi 0-determinateness imply (WD1) or (WD3).

Proof. Let (A, B) be a $2\times\infty$ bimatrix game.

(a). Let (A, B) be 0-determined. Then there is a $p\in\Delta_2$ such that $\lambda(p)=0$. Suppose $p\in UB(C)$. Then for every $\varepsilon_2>0$ there either is a $q\in\Delta_\infty$ such that $Bq\in O_{p,\varepsilon_2}$ and $e_1Aq=e_2Aq$ or there are $q^1, q^2\in\Delta_\infty$ such that $Bq^1, Bq^2\in O_{p,\varepsilon_2}$, $e_1Aq^1>e_2Aq^1$ and $e_1Aq^2>e_2Aq^1$. In the latter case we can find $\tilde{q}\in\operatorname{conv}(\{q^1, q^2\})$ with $e_1A\tilde{q}=e_2A\tilde{q}$. Since $O_{p.\varepsilon_2}$ is convex, also $B\tilde{q}\in O_{p,\varepsilon_2}$. So the latter case implies the first one. Hence for every $\varepsilon_2>0$ we can find a $q\in\Delta_\infty$ such that $pBq\geq\sup_{y\in\Delta_\infty}pBy-\varepsilon_2$ and $e_1Aq=e_2Aq$. The last equality implies that for every $\varepsilon_1>0$ we have that $pAq>\sup_{x\in\Delta_2}xAq-\varepsilon_1$. Hence (A, B) has the property (WD1). Similarly one shows that (A, B) has the property (WD3) if $p\in NUB(C)$.

(b). Let (A, B) be 1-determined. Then $\lambda(e_1)=1$. Suppose $e_1\in NUB(C)$. Then there is a $\bar{k}\in\mathbb{R}$ such that $e_1Aq>e_2Aq$ for all $q\in\Delta_\infty$ with $Bq\in G_{e_1,\bar{k}}$. Let $\varepsilon_1>0$ and $k_2\in\mathbb{R}$. If $k_2\geq\bar{k}$, then take $q\in\Delta_\infty$ such that $Bq\in G_{e_1,k_2}\subset G_{e_1,\bar{k}}$. If $k_2<\bar{k}$, then take $q\in\Delta_\infty$ such that $Bq\in G_{e_1,\bar{k}}\subset G_{e_1,k_2}$. In both cases $e_1Aq>e_2Aq$, and hence $e_1Aq\geq\sup_{x\in\Delta_2}xAq-\varepsilon_1$. Moreover, since also in both cases $Bq\in G_{e_1,k_2}$ we have $e_1Bq\geq k_2$. Hence (e_1, q) is an (ε_1, k_2)-equilibrium in both cases. Since $\varepsilon_1>0$ and $k_2\in\mathbb{R}$ were arbitrary, (A, B) has the property (WD3). Similarly one shows that (A, B) has the property (WD1) if $e_1\in UB(C)$.

(c). Similar to (b) one shows that 2-determinateness implies (WD1) ($e_2\in UB(C)$) or (WD3) ($e_2\in NUB(C)$).

(d). Let (A, B) be quasi 0-determined. Let $\varepsilon_1, \varepsilon_2>0$. By definition $UB(C)\neq\emptyset$ and we can find a pair $(p, q)\in\Delta_2\times\Delta_\infty$ such that $p\in UB(C)$, $Bq\in O_{p,\varepsilon_2}$ and $e_1Aq=e_2Aq$. This implies that (p, q) is an $(\varepsilon_1, \varepsilon_2)$-equilibrium. Hence quasi 0-determinateness implies (WD1).

In order to prove the final result of this section, we need two lemmas.

Lemma 4.1. Let (A, B) be a $2\times\infty$ bimatrix game and let $C:=\{Bq\mid q\in\Delta_\infty\}$. Suppose $UB(C)=\emptyset$. Then (A, B) is either 1- or 2-determined if (A, B) is *not* 0-determined.

Proof. Let (A, B) not be 0-determined. Then $\lambda(p)\in\{1, 2\}$ for all $p\in\Delta_2=NUB(C)$. Suppose $p^1, p^2\in\Delta_2$ exist such that $\lambda(p^1)=1$ and $\lambda(p^2)=2$. Then $k_1, k_2\in\mathbb{R}$ exist such that $e_1Aq^1>e_2Aq^1$ for all $q^1\in\Delta_\infty$ with $Bq^1\in G_{p^1,k_1}$ and $e_1Aq^2<e_2Aq^2$ for all $q^2\in\Delta_\infty$ with $Bq^2\in G_{p^2,k_2}$. This implies $C\cap\{x\in\mathbb{R}^2\mid p^1\cdot x\geq k_1 \text{ and } p^2\cdot x\geq k_2\}=\emptyset$. Applying lemma 3.1 we obtain a $p\in\operatorname{conv}(\{p^1, p^2\})$ such that $px\leq py$ for all $x\in C$ and all $y\in\mathbb{R}^m$ satisfying $p^1y\geq k_1$ and $p^2y\geq k_2$. Let y^0 be the solution of the system of equations $p^1y^0=k_1$ and $p^2y^0=k_2$. Then $px\leq py^0$ for all $x\in C$. Hence $\sup_{x\in C}px<\infty$. This contradicts $UB(C)=\emptyset$. So either $\lambda(p)=1$ for all $p\in\Delta_2$ or $\lambda(p)=2$ for all $p\in\Delta_2$, which proves the lemma. □

Lemma 4.2. Let (A, B) be a $2\times\infty$ bimatrix game and let $C:=\{Bq\mid q\in\Delta_\infty\}$. Suppose $UB(C)\neq\emptyset$. Then λ is constant on $UB(C)$, if (A, B) is neither 0- nor quasi 0-determined.

Proof. Let (A, B) be neither 0- nor quasi 0-determined. In view of the remark above concerning the negation of quasi 0-determinateness there exists an $\varepsilon_0>0$ such that for every pair $(p, q)\in UB(C)\times\Delta_\infty$ with $Bq\in O_{p,\varepsilon_0}$ we have $e_1Aq\neq e_2Aq$. Using the convexity of O_{p,ε_0} for any $p\in UB(C)$ we find that for every $p\in UB(C)$, $\lambda(p)=1$ implies $e_1Aq>e_2Aq$ for all $q\in\Delta_\infty$ with $Bq\in O_{p,\varepsilon_0}$ and $\lambda(p)=2$ implies $e_1Aq<e_2Aq$ for all $q\in\Delta_\infty$ with $Bq\in O_{p,\varepsilon_0}$.

Take $p^1, p^2\in UB(C)$. Let $p(1), p(2), \ldots$ be a sequence of elements in $\mathrm{conv}(\{p^1, p^2\})$ with $\lambda(p(n))=1$ for all $n\in\mathbb{N}$, which converges to $\hat{p}$. Clearly $\hat{p}\in\mathrm{conv}(\{p^1, p^2\})$. We show that $\lambda(\hat{p})=1$. By proposition 3.2 there is a finite subset S of C such that $S\cap O_{p,\varepsilon_0}\neq\emptyset$ for all $p\in\mathrm{conv}(\{p^1, p^2\})$. So for each $n\in\mathbb{N}$ we can take $q(n)\in\Delta_\infty$ such that $Bq(n)\in S\cap O_{p(n),\frac{1}{2}\varepsilon_0}$. Consequently $e_1Aq(n)>e_2Aq(n)$ for every n. Since S is a finite set we may assume, without loss of generality, that $q(n)=\hat{q}$ for all n. Take $q^0\in\Delta_\infty$ such that $Bq^0\in O_{\hat{p},\frac{1}{2}\varepsilon_0}$. Then for each $n\in\mathbb{N}$ we have $p(n)B\hat{q}\geq\sup_{x\in C}p(n)\cdot x-\frac{1}{2}\varepsilon_0\geq p(n)Bq^0-\frac{1}{2}\varepsilon_0$. This implies that $\hat{p}B\hat{q}\geq\hat{p}Bq_0-\frac{1}{2}\varepsilon_0\geq\sup_{x\in C}\hat{p}\cdot x-\frac{1}{2}\varepsilon_0-\frac{1}{2}\varepsilon_0$, so that $Bq\in O_{\hat{p},\varepsilon_0}$. Since $e_1A\hat{q}>e_2A\hat{q}$, this implies $e_1Aq>e_2Aq$ for all $q\in\Delta_\infty$ with $Bq\in O_{\hat{p},\varepsilon_0}$. Hence $\lambda(\hat{p})=1$. So the set of all $p\in\mathrm{conv}(\{p^1, p^2\})$ having $\lambda(p)=1$ is closed. Similarly the set of $p\in\mathrm{conv}(\{p^1, p^2\})$ having $\lambda(p)=2$ is closed. Since (A, B) is not 0-determined, $\mathrm{conv}(\{p^1, p^2\})$ is the union of these two sets. But then either $\lambda(p)=1$ or $\lambda(p)=2$ for all $p\in\mathrm{conv}(\{p^1, p^2\})$. □

We are now in a position to prove

Theorem 4.1. Every $2\times\infty$ bimatrix game is 0-, 1-, 2- or quasi 0-determined.

Proof. Let (A, B) be a $2\times\infty$ bimatrix game and let $C:=\{Bq\mid q\in\Delta_\infty\}$. Suppose (A, B) is neither 0- nor quasi 0-determined. We show that (A, B) is either 1- or 2-determined. In view of lemmas 4.1 and 4.2 we may suppose $UB(C)\neq\emptyset$ and $NUB(C)\neq\emptyset$. By lemma 4.2 λ is constant on $UB(C)$ and since (A, B) is not 0-determined, we can assume without loss of generality that $\lambda(p)=1$ for all $p\in UB(C)$. Since (A, B) is not quasi 0-determined we can find an $\varepsilon_0>0$ such that for all $p\in UB(C)$ and $q\in\Delta_\infty$ with $Bq\in O_{p,\varepsilon_0}$ we have $e_1Aq>e_2Aq$. Now take $p\in NUB(C)$ and $k\in\mathbb{R}$. Then, by proposition 3.1, there is a $p^1\in UB(C)$ such that $G_{p,k}\cap O_{p^1,\varepsilon_0}\neq\emptyset$. Take $q\in\Delta_\infty$ such that $Bq\in G_{p,k}\cap O_{p^1,\varepsilon_0}$. Since $e_1Aq>e_2Aq$ and since (A, B) is not 0-determined, this implies that $\lambda(p)=1$. So we have shown that $\lambda(p)=1$ for all $p\in\Delta_2$ and hence that (A, B) is 1-determined.

Similarly one shows that (A, B) is 2-determined if $\lambda(p)=2$ for all $p\in UB(C)$. □

As we have mentioned before we have in view of proposition 4.1.

Corollary 4.1. Every $2\times\infty$ bimatrix game has the property (WD1) or (WD3).

5 Remarks

Carefully looking at the proof of propositon 4.1 one will find that a $2 \times \infty$ bimatrix game actually has stronger properties than (WD1) or (WD3), namely for every $2 \times \infty$ bimatrix game we have:

(i) there is a $(0, \varepsilon)$-equilibrium for every $\varepsilon > 0$ or
(ii) there is a $(0, k)$-equilibrium for every $k \in \mathbb{R}$.

For $m \times \infty$ bimatrix games where $m \geq 3$, our labeling method fails. The actual problem is the following: in the proof of proposition 4.1 we used that, if there are $q^1, q^2 \in \Delta_\infty$ such that $e_1 A q^1 > e_2 A q^1$ and $e_1 A q^2 < e_2 A q^2$, then there is a $q \in \operatorname{conv}(\{q^1, q^2\})$ such that $e_1 A q = e_2 A q$. This property fails to hold already for $m = 3$. Let (A, B) be a $3 \times \infty$ bimatrix game such that the first column of A is $\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$ and the second is $\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$. Let $q^1 = (1, 0, 0, \ldots)$ and $q^2 = (0, 1, 0, \ldots)$. Then $e_1 A q^1 > e_3 A q^1 > e_2 A q^1$ and $e_2 A q^2 > e_3 A q^2 > e_1 A q^2$, but there is no $q \in \operatorname{conv}(\{q^1, q^2\})$ with $e_1 A q^2 = e_2 A q^2 \geq e_3 A q^2$. A similar problem occurred in Borm, Gijsberts and Tijs (1988).

Finally we remark that the results stated in this paper also hold for mixed extensions of two-person games $\langle X, Y, k_1, k_2 \rangle$, where $|X| = 2$ and Y is arbitrary.

References

Borm P, Gijsberts A, Tijs SH (1988) A geometric-combinatorial approach to bimatrix games. Methods Oper. Res. 59:199–209.
Lucchetti R, Patrone F, Tijs SH (1986) Determinateness of two-person games. Bollettino U.M.I. 6:907–924.
Nash JF (1951) Non-cooperative games. Ann. of Math. 54:286–295.
Tijs SH (1975) Semi-infinite and infinite matrix games and bimatrix games. PhD Dissertation, Dept. of Math., University of Nijmegen, The Netherlands.
Tijs SH (1977) ε-Equilibrium points for two-person games. Methods Oper. Res. 26:755–766.
Tijs SH (1981) Nash equilibria for non-cooperative n-person games in normal form. SIAM Review 23:225–237.
Von Neumann J (1928) Zur Theorie der Gesellschaftsspiele. Math. Ann. 100:295–320.
Wald A (1945) Generalization of a theorem by von Neumann concerning zero sum two-person games. Ann. of Math. 46:281–286.

Received November 1991
Revised version November 1992